What is the fundamental group?

a.k.a "Why is algebra important for topology?"

Robin Truax

October 2021

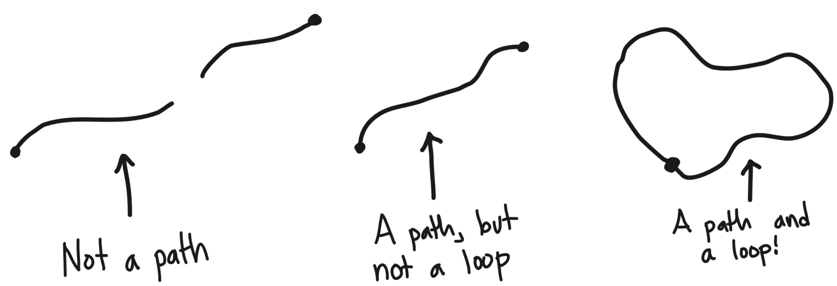
1 Introduction

This is a casual introduction to the idea of "the fundamental group" and its applications. Not only will this help explain the name of my favorite Discord server for Stanford math students, but it will also help illustrate why algebra (group theory, category theory, etc.) is so important for topology.

Prerequisites: Basic knowledge about groups.

2 The Fundamental Group

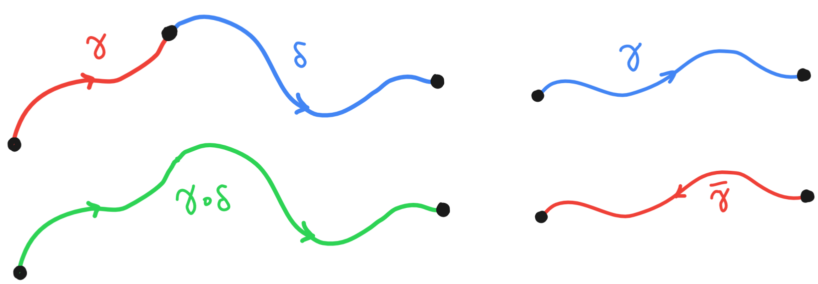
Formally, a ***path*** in a topological space X is a continuous map $\gamma : [0,1] \to X$. The ***endpoints*** of γ are $\gamma(0)$ and $\gamma(1)$. A path γ is called a ***loop*** if its endpoints are equal; that is, if $\gamma(0) = \gamma(1)$.



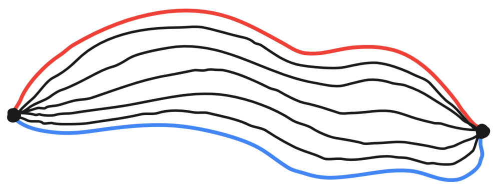
Now, there exist some important operations on paths. For example, suppose γ is a path with endpoints x_0 and x_1, and δ is a path with endpoints x_1 and x_2. Then $\gamma \cdot \delta$, the ***product of*** γ ***and*** δ, is a path from x_0 and x_1 that "does γ and then does δ". Formally, we define

$$(\gamma \cdot \delta)(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \delta(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Another important operation is called the ***reverse***, and takes just one path. The reverse of γ, denoted $\overline{\gamma}$ is the path that "does γ backwards". Formally, $\overline{\gamma}(t) = \gamma(1-t)$. If you want, try checking that both $\gamma \cdot \delta$ and $\overline{\gamma}$ are continuous. Following are some visual examples.



In topology, often spaces are considered the same if one can be continuously deformed into the other and vice-versa. The same holds for paths: two paths γ and δ with the same endpoints are ***path-homotopic*** if γ can be continuously deformed into δ while fixing the endpoints.



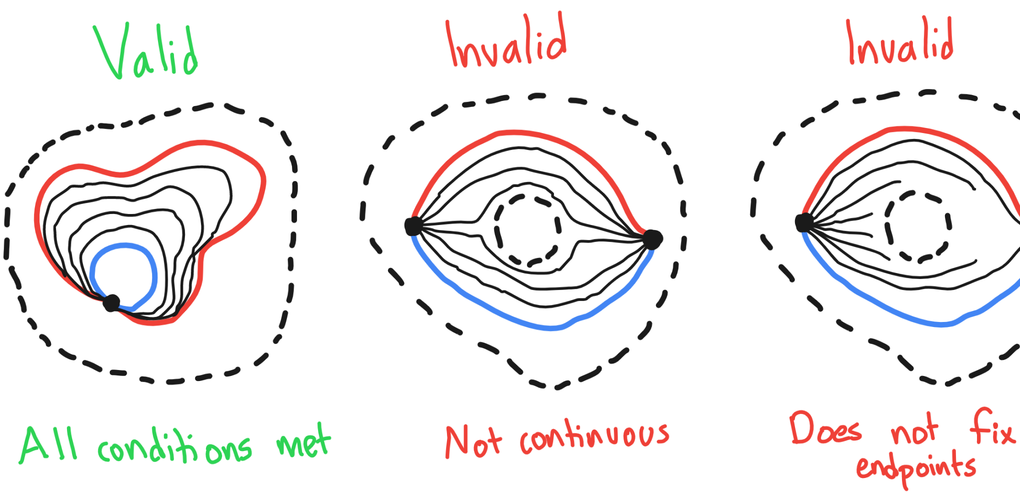
Formally, a ***path homotopy*** from γ to δ is a continuous map $F : [0,1] \times [0,1] \to X$ such that

1. $F(x, 0) = \gamma(x)$ and $F(x, 1) = \delta(x)$ for all x,
2. $\gamma(0) = F(0, t) = \delta(0)$ and $\gamma(1) = F(1, t) = \delta(1)$ for all t.

We say that γ and δ are path-homotopic if there exists a path homotopy between them.

Now, the formal definition of path homotopies can be a bit intimidating, so let's break it down a bit. What's actually going on is that we have a family of paths f_t, given by $f_t(x) = F(x, t)$, for all $t \in [0, 1]$. This family of paths varies continuously (since F is continuous in both x and t) between $f_0 = \gamma$ and $f_1 = \delta$ (this is the first condition) and shares the same endpoints for any t (this is the second condition).

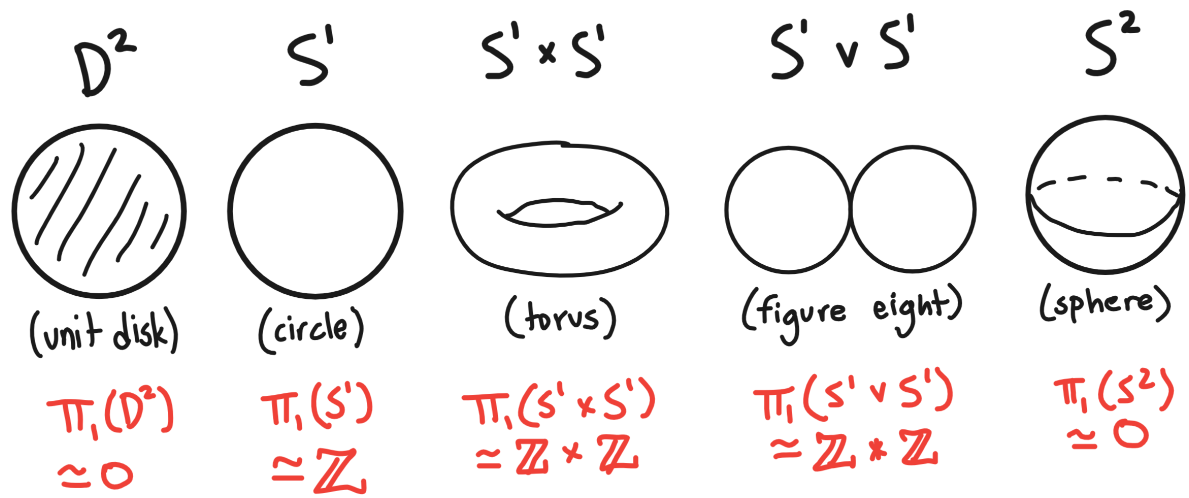
Here are some examples of valid and invalid path-homotopies.



Suppose we fix points $x_0, x_1 \in X$. Then path-homotopies form an equivalence relation on the set of all paths between x_0 and x_1; that is, $\gamma \sim \delta$ are equivalent if there exists a path-homotopy from γ to δ. Check yourself that this truly is an equivalence relation: in particular, transitivity raises an interesting problem. Under this relation, the equivalence class of γ is called the ***path-homotopy class*** of γ and is denoted $[\gamma]$.

Now, we're ready to define the fundamental group. The ***fundamental group of*** X ***at*** x_0, denoted $\pi_1(X, x_0)$, is the set of all homotopy classes of loops starting and ending at x_0. The group operation is given by the product $\cdot$, the identity is the constant map, and the inverse of a loop γ is $\overline{\gamma}$. I leave it to you to check that $\pi_1(X, x_0)$ satisfies the group axioms; when you do, remember the elements are not paths, but *classes* of paths.

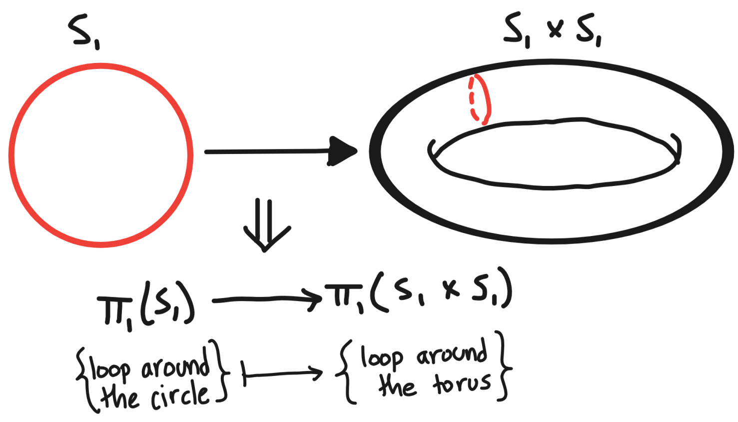
Following are example computations of the fundamental group (the proofs can be found in Hatcher's *Algebraic Topology*, available freely online here). The computations are a lot of work, so we don't do them here.



Finally, let's explore how the fundamental group π_1 is a ***functor***. Functors are an idea from category theory, but the idea is simple enough that no prior experience with category theory is necessary. Roughly speaking, functors carry us from one "world of math" to another "world of math". Notice that I do not simply say that functors transform one type of mathematical object into another type of mathematical object; functors do more than that. The true power of functors come from the fact that they *also* transform maps between one type of object into maps between another type of object.

Let's explore what it means for the fundamental group to be a functor in more detail; this will help us understand the concept of a functor more generally. Now, the type of object that π_1 "accepts" is called a ***based topological space***. No, not that kind of "based"; a based topological space is just a pair (X, x_0), where x_0 is a point in X which we call the ***basepoint***. For us, it's the point which all our loops begin and end at. The type of object which π_1 "returns" is just an ordinary group $\pi_1(X, x_0)$.

Now, let's consider how the fundamental group can transform maps. Suppose that $\phi : (X, x_0) \to (Y, y_0)$ is a morphism of based topological spaces (this is just a fancy way to say that $\phi : X \to Y$ is continuous map such that $\phi(x_0) = y_0$). Then ϕ *induces* a map $\pi_1(\phi) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $[\gamma] \mapsto [\phi \circ \gamma]$. Let us consider a visual example to illustrate this concept:

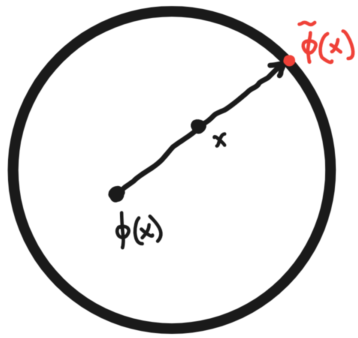


Here's the special thing: this map $\pi_1(\phi)$ isn't just a function, but a *group homomorphism*. That is, $\pi_1(\phi)$ transforms morphisms of based topological spaces to morphisms of groups! This transformation of morphisms also has some special properties: $\pi_1(\phi \circ \psi) = \pi_1(\phi) \circ \pi_1(\psi)$ (that is, π_1 respects composition) and $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)}$ (π_1 takes the identity map to the identity map). Try checking this yourself!

These properties might seem innocent, but they're incredibly powerful, and to show how, we're going to prove Brouwer's Fixed Point Theorem using the fundamental group.

Theorem 1 (Brouwer's Fixed Point Theorem)**.** *Any continuous map ϕ from the unit disk D^2 to itself has a fixed point; that is, there exists some $x \in D^2$ such that $\phi(x) = x$.*

Proof. Assume that the theorem is false; that is, for the sake of contradiction take a map $\phi : D^2 \to D^2$ such that $\phi(x) \neq x$ for all x. Then, we may define a continuous map $\widetilde{\phi} : D^2 \to S^1$ given by sending x to the point on the boundary of D^2 on the ray starting at $\phi(x)$ and going through x. Visually, $\widetilde{\phi}$ is defined as so:



Notice that $\widetilde{\phi}(x) = x$ for every $x \in S^1$. That is, $\widetilde{\phi(x)}$ fixes the boundary circle S^1. I claim that this gives us a contradiction. To see why, let $\iota : S^1 \to D^2$ be the natural embedding map (taking the boundary circle to itself within the unit disk). Then, $\widetilde{\phi} \circ \iota$ is the identity map on S^1. In category theory, we visually represent this relationship with a "commutative diagram", as so:

$$S^1 \xrightarrow{\iota} D^2 \xrightarrow{\widetilde{\phi}} S^1 \qquad \mathrm{id}_{S_1}$$

But consider what happens if we apply our functor π_1 to this diagram. As mentioned before, $\pi_1(D^2)$ is the trivial group 0, whereas $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$. Hence we have a commutative diagram as so:

$$\mathbb{Z} \xrightarrow{\pi_1(\iota)} 0 \xrightarrow{\pi_1(\widetilde{\phi})} \mathbb{Z} \qquad \mathrm{id}_{\mathbb{Z}}$$

But this is impossible! How can a homomorphism from $\mathbb{Z}$ onto the trivial group back to $\mathbb{Z}$ be the identity; clearly, it cannot be either injective or surjective (since all of $\mathbb{Z}$ is mapped onto the only element in the trivial group, which is mapped to the identity element in $\mathbb{Z}$). Hence we have the desired contradiction. ☐

The applications of Brouwer's Fixed-Point Theorem are numerous and beautiful. For example, it implies that if one places any map within the bounds of an identical but larger map, there will be at least one spot which is the same place on both maps, however the smaller map is oriented. More interestingly, it can be used to show that the game of Hex always a winner (see this article). This connection between topology and game theory is surprising and beautiful.