What Distinguishes Category Theory?
Maps vs. Elements

Robin Truax

March 17th, 2021

In this paper, I'm going to give two competing (and equivalent) definitions of *quotients of vector spaces* to illustrate why the perspective offered by category theory is so interesting. We will begin with the standard perspective where **elements** matter, and move on to the category-theoretic perspective where **maps** matter.

For this paper, let V and W be vector spaces (link to general definition) over a field F (such as $\mathbb{R}$ or $\mathbb{C}$).

1 The Elements Matter

Definition 1. Suppose $T : V \to W$ is a linear transformation. Then the *kernel* of T, denoted $\ker T$, is

$$\{v \in V \mid T(v) = 0\}.$$

The kernel of T is also called *the null space* of T, and it is relatively easy to prove that it is a subspace of V.

Definition 2. Suppose N is a subspace of V. Then, for any $x \in V$, the *coset of* x, denoted $[x]$ or $x + N$, is

$$\{x + n \mid n \in N\}.$$

Theorem 1. *Any two cosets are either disjoint or identical. More precisely, if $x + N$ and $y + N$ are not disjoint, then they are equal as sets.*

Proof. Suppose $z \in x + N$ and $z \in y + N$. Then $z = x + n_1 = y + n_2$ for $n_1, n_2 \in N$. Yet then, for any $n \in N$, by the properties of subspaces $n_1 - n_2 + n \in N$, so

$$y + n = x + (n_1 - n_2 + n) \in x + N.$$

This proves that $y + N \subseteq x + N$, and identical reasoning proves that $x + N \subseteq y + N$, so they are equal. $\square$

Definition 3. If N is a subspace of V, then the *quotient space* V/N is defined to be the set of cosets of N. In particular, any element of V/N is of the form $x + N$ for some vector $x \in N$. To make V/N a vector space over F, we define vector addition by

$$(x + N) + (y + N) = (x + y) + N.$$

We also define scalar multiplication by

$$c(x + N) = cx + N.$$

Problem 1. Prove that the above definitions are "well-defined" on the cosets of N. More precisely, prove that if $x + N = x' + N$ and $y + N = y' + N$, then

$$(x + N) + (y + N) = (x' + N) + (y' + N) \textbf{ and } c(x + N) = c(x' + N).$$

In doing this, it may be helpful to prove and use the following fact: $x + N = x' + N$ if and only if $x - x' \in N$.

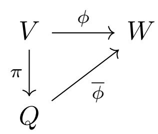
Problem 2. Suppose V is the $\mathbb{R}$-vector space $\mathbb{R}^4$ and N is the subspace spanned by the vectors $\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^\top$ and $\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}^\top$ (under the standard basis). Find a basis for V/N and prove that $V/N \simeq \mathbb{R}^2$.

2 The Maps Matter

In this section "map" is short for "linear transformation". Now, there is a way to define subspaces and kernels using maps alone, but to focus on quotient spaces, we use the classical definitions of these objects.

Definition 4. Suppose N is a subspace of V. Then a *quotient of V by N* is a pair (Q, π) where Q is a vector space and π is a surjective map $V \to Q$ such that $N \subseteq \ker \pi$ and the following property holds:

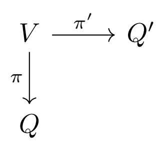
Given any map $\phi : V \to W$ such that $N \subseteq \ker \phi$, there is a unique map $\overline{\phi}$ such that



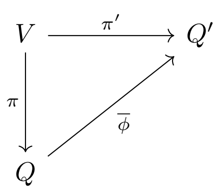
commutes; that is, $\overline{\phi} \circ \pi = \phi$. This is called the *universal property* of quotients.

Theorem 2. *Suppose N is a subspace of V. Also suppose (Q, π) and (Q', π') are two quotients of V by N. Then there is an isomorphism $\iota : Q \xrightarrow{\sim} Q'$ with $\iota \circ \pi = \pi'$.*

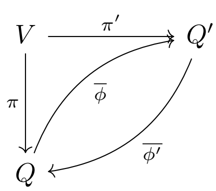
Proof. Consider the following diagram



In particular, since $N \subseteq \ker \pi'$, the universal property of quotients tells us that there is a unique map $\overline{\phi} : Q \to Q'$ such that the following diagram commutes:



By the same logic, there exists a unique map $\overline{\phi'} : Q' \to Q$ such that the following diagram commutes:



Now we simply have to prove that $\overline{\phi}$ and $\overline{\phi'}$ are inverses. To do this, notice that because $N \subseteq \ker \pi$, the universal property of quotients tells us that there is a *unique* map $\overline{\pi} : Q \to Q$ such that $\overline{\pi} \circ \pi = \pi$. Now, the identity map $\mathrm{id}_Q : Q \to Q$ satisfies $\mathrm{id}_Q \circ \pi = \pi$, but since the above diagram commutes, we also find that $(\overline{\phi'} \circ \overline{\phi}) \circ \pi = \pi$ too. Since $\overline{\pi}$ is unique, we may indeed conclude that $\overline{\phi'} \circ \overline{\phi} = \mathrm{id}_Q$. By identical logic on the uniqueness of $\overline{\pi'}$, we may conclude that $\overline{\phi} \circ \overline{\phi'} = \mathrm{id}_{Q'}$, so indeed $\overline{\phi}$ and $\overline{\phi'}$ are inverses. Therefore, they are both isomorphisms, and in particular $\overline{\phi}$ is an isomorphism $Q \xrightarrow{\sim} Q'$ with $\overline{\phi} \circ \pi = \pi'$. ☐

Corollary 2.1. *Up to isomorphism, there is a* unique *quotient of V by N. We denote this quotient V/N.*

Problem 3. Repeat Problem 2 using the "map-perspective" definition of quotients.

3 It's All the Same (up to Isomorphism)

Theorem 3. *The definition of V/N using cosets satisfies the universal property of a quotient of V by N.*

Proof. Let $\pi : V \to V/N$ be the map given by sending x to $x + N$. Clearly, this map is surjective and $\ker \pi = N$. Next, suppose that $\phi : V \to W$ is a linear transformation such that $N \subseteq \ker \phi$. We must demonstrate that there is a unique linear transformation $\overline{\phi} : V/N \to W$ satisfying $\overline{\phi} \circ \pi = \phi$.

Now, if $\overline{\phi} \circ \pi = \phi$, then for any $x \in V$, $\overline{\phi}(\pi(x)) = \phi(x)$, so $\overline{\phi}(x + N) = \phi(x)$. This tells us that the behavior of $\overline{\phi}$ is completely determined, so this is the *only valid option* for $\overline{\phi}$. Thus, if we show that $\overline{\phi}$ is indeed a linear transformation, then it is the unique linear transformation with $\overline{\phi} \circ \pi = \phi$, as desired.

There is some nuance here: it's not entirely obvious that our definition is well-defined. Precisely, if $x + N = x' + N$, how can we be sure that $\phi(x) = \phi(x')$ so $\overline{\phi}(x + N) = \overline{\phi}(x' + N)$? To prove this, recall that $x + N = x' + N$ if and only if $x - x' \in N$. But since $N \subseteq \ker \phi$, this implies that $\phi(x - x') = 0$. But then, because ϕ is a linear transformation, $\phi(x) - \phi(x') = 0$, and indeed $\phi(x) = \phi(x')$.

Now that we know $\overline{\phi}$ is well-defined, it suffices to prove that it is a linear transformation. Yet this is simple if we recall that ϕ is a linear transformation. That is, if $c, d \in F$ and $x, y \in V$, then

$$\overline{\phi}(c(x + N) + d(y + N)) = \overline{\phi}((cx + dy) + N) = \phi(cx + dy) = c\phi(x) + d\phi(y) = c\overline{\phi}(x + N) + d\overline{\phi}(y + N).$$

Thus $\overline{\phi}$ is linear, so we are done. □

Corollary 3.1. *Both definitions of quotient spaces (classical and map-based) are isomorphic by Theorem* 2.