Functors, Natural Transformations, and Adjoints
The Nuts and Bolts

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Contents

1 Functors

Definition 1 (Covariant Functor). A *(covariant) functor* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is a pair of maps $\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$ and $\mathrm{Mor}(\mathbf{C}) \to \mathrm{Mor}(\mathbf{D})$ such that the following conditions hold:

(1) $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$ for all objects $X \in \mathbf{C}$.

(2) Given any pair of morphisms $f : X \to Y$ and $g : Y \to Z$, $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

Definition 2 (Contravariant Functor). A *contravariant functor* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is a pair of maps $\mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$ and $\mathrm{Mor}(\mathbf{C}) \to \mathrm{Mor}(\mathbf{D})$ such that the following conditions hold:

(1) $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$ for all objects $X \in \mathbf{C}$.

(2) Given any pair of morphisms $f : X \to Y$ and $g : Y \to Z$, $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

2 Natural Transformations

Definition 3 (Natural Transformation). Given two functors $\mathcal{F}, \mathcal{G} : \mathbf{C} \rightrightarrows \mathbf{D}$, a *natural transformation (of covariant functors)* $\eta : \mathcal{F} \to \mathcal{G}$ is a family of morphisms $\eta_X : \mathcal{F}(X) \to \mathcal{G}(X)$ for each X in $\mathbf{C}$, such that for any $f : X \to Y$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\
{\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\
\mathcal{G}(X) & \xrightarrow[\mathcal{G}(f)]{} & \mathcal{G}(Y)
\end{array}
$$

Similarly, a *natural transformation of contravariant functors* can be defined by switching the direction of the arrows $\mathcal{F}(f)$ and $\mathcal{G}(f)$, and keeping all the other requirements the same.

If η is a natural transformation, we say that the family of morphisms η_X are *natural in X*. If all of the arrows η_X are furthermore isomorphisms, then we say that η is a *natural isomorphism* and write $\mathcal{F} \simeq \mathcal{G}$.

2.1 Examples

Example 1 (Abelianization). Define a functor Ab : $\mathbf{Grp} \to \mathbf{Grp}$ as follows: for any group $G \in \mathbf{Grp}$, $\mathrm{Ab}(G) = G/[G,G]$ is the abelianization of G. Given any map $f : G \to H$ in $\mathbf{Grp}$, $\mathrm{Ab}(f)$ is defined as so: the composition $G \xrightarrow{f} H \xrightarrow{\pi_H} H/[H,H]$ contains $[G,G]$ in its kernel (since $H/[H,H]$ is abelian), so there exists a map $G/[G,G] \to H/[H,H]$, defined to be $\mathrm{Ab}(f)$, such that the following diagram commutes:

$$
\begin{array}{ccc}
G & \xrightarrow{\ \ f\ \ } & H \\
{\scriptstyle \pi_G}\downarrow & & \downarrow{\scriptstyle \pi_H} \\
G/[G,G] & \xrightarrow[\mathrm{Ab}(f)]{} & H/[H,H]
\end{array}
$$

The above diagram also shows that π is a natural transformation from id : $\mathbf{Grp} \to \mathbf{Grp}$ to Ab : $\mathbf{Grp} \to \mathbf{Grp}$.

Example 2 (Double Dual). Let k be a field, and Vect_k be the category of vector spaces over k. Define a functor $D : \mathrm{Vect}_k \to \mathrm{Vect}_k$ as follows:

(1) Given any object V, $D(V)$ is the double dual V^{**} of V.

(2) Given any map $f : V \to W$, $D(f) = f^{**}$ is the map constructed as follows: first, we define $f^* : W^* \to V^*$ by $(\varphi : W \to k) \mapsto (\varphi \circ f : V \to k)$. Then we define $D(f) = f^{**} : V^{**} \to W^{**}$ by $\varphi : V^* \to k) \mapsto (\varphi \circ f^* : W^* \to k)$; that is, a map $\varphi : \mathrm{Hom}(V,k) \to k$ is sent to the map $f^{**}(\varphi) : \mathrm{Hom}(W,k) \to k$ given by $T \mapsto \varphi(T \circ f)$.

One may check that the axioms for a functor holds. Furthermore, I claim that the family of morphisms $\iota_V : V \to V^{**}$ given by sending v to the map $\mathrm{Hom}(V,k) \to k$ given by $\varphi \mapsto \varphi(v)$ forms a natural transformation from id to D. For this, one must check that the following diagram commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{\ \ f\ \ } & W \\
{\scriptstyle \iota_V}\downarrow & & \downarrow{\scriptstyle \iota_W} \\
V^{**} & \xrightarrow[f^{**}]{} & W^{**}
\end{array}
$$

For this, fix $v \in V$. Then $\iota_W(f(V))$ is the map is the map $\mathrm{Hom}(W,k) \to k$ given by sending $T : W \to k$ to $T(f(v))$. On the other hand, $\iota_V(v)$ is the map $\mathrm{Hom}(V,k) \to k$ given by sending $T : V \to k$ to $T(v)$. Then $f^{**}(\iota_V(v))$ is the map $\mathrm{Hom}(W,k) \to k$ given by sending $T : W \to k$ to $(\iota_V(v))(T \circ f)$, but the latter is just $(T \circ f)(v) = T(f(v))$, exactly the same as $\iota_W(f(v))$. Hence $\iota_W(f(v)) = f^* * (\iota_V(v)) = T \mapsto T(f(v))$. Therefore, $\iota_W \circ f = f^{**} \circ \iota_V$, exactly as desired.

2.2 Equivalence of Categories

Definition 4 (Isomorphism of Categories). An *isomorphism of categories* is a pair of functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ such that $\mathcal{G}\mathcal{F} = \mathrm{id}_{\mathbf{C}}$ and $\mathcal{F}\mathcal{G} = \mathrm{id}_{\mathbf{D}}$. This requirement can occur, but it is usually too strict.

Definition 5 (Equivalence of Categories). An *equivalence of categories* is a pair of functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ such that $\mathcal{G}\mathcal{F} \simeq \mathrm{id}_{\mathbf{C}}$ and $\mathcal{F}\mathcal{G} \simeq \mathrm{id}_{\mathbf{D}}$. This is a far more important requirement.

To give an idea of why Definition 5 is more important than Definition 4, consider the case of $\mathbf{Sets}_{\mathrm{fin}}$, the category of all finite sets and maps between them. In some sense, all the data of this category is encoded in the category of $\mathbf{Ord}_{\mathrm{fin}}$, the category of all finite ordinal numbers with the usual set-theoretic maps between them. That is, given a set S with 2 elements and a set T with 3 elements, only the numbers 2 and 3 are needed to characterize $\mathrm{Hom}(S,T)$.

Yet notice that there exists no isomorphism of categories between $\mathbf{Sets}_{\mathrm{fin}}$ and $\mathbf{Ord}_{\mathrm{fin}}$ (for cardinality reasons alone), but in fact there is an equivalence of categories between $\mathbf{Sets}_{\mathrm{fin}}$ and $\mathbf{Ord}_{\mathrm{fin}}$; namely the obvious one which sends any ordinal to itself, and any set to the unique ordinal of the same size.

Theorem 1 (Equivalence Criterion). *A functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ gives an equivalence of categories iff it is*

(1) *Full (that is, for any objects $X, Y \in \mathbf{C}$, the map $\mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is surjective).*

(2) *Faithful (that is, for any objects $X, Y \in \mathbf{C}$, the map $\mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is injective).*

(3) *Essentially surjective (given any object $Y \in \mathbf{D}$, there exists an object $X \in \mathbf{C}$ such that $\mathcal{F}(X) \simeq Y$).*

Proof. Suppose that $\mathcal{F}$ is an equivalence of categories $\mathbf{C} \to \mathbf{D}$. Then there exists another function $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\alpha : \mathrm{id}_{\mathbf{C}} \xrightarrow{\sim} \mathcal{G}\mathcal{F}$ and $\beta : \mathrm{id}_{\mathbf{D}} \xrightarrow{\sim} \mathcal{F}\mathcal{G}$.

Now suppose that $f, f' : X \rightrightarrows Y$ are such that $\mathcal{F}(f) = \mathcal{F}(f')$. Then $\mathcal{G}\mathcal{F}(f) = \mathcal{G}\mathcal{F}(f')$, so by the naturality of α, the following diagram commutes (note that the horizontal arrows are isomorphisms):

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;\alpha_X\;\;} & \mathcal{G}\mathcal{F}(X) \\[2pt]
{\scriptstyle f}\Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle \mathcal{G}\mathcal{F}(f)=\mathcal{G}\mathcal{F}(f')} \\[6pt]
Y & \xrightarrow[\;\;\alpha_Y\;\;]{\sim} & \mathcal{G}\mathcal{F}(Y)
\end{array}
$$

Yet then $f = \alpha_Y^{-1} \circ \mathcal{G}\mathcal{F}(f) \circ \alpha_X = \alpha_Y^{-1} \circ \mathcal{G}\mathcal{F}(f') \circ \alpha_X = f'$, so in fact $\mathcal{F}$ is faithful. Notice that identical logic demonstrates that $\mathcal{G}$ is also faithful; we will use this fact later.

Now take a map $g : \mathcal{F}(X) \to \mathcal{F}(Y)$; we will demonstrate that $g = \mathcal{F}(f)$ for some $f : X \to Y$. Namely, inspired by the above commutative diagram, define $f : X \to Y$ to be $\alpha_Y^{-1} \circ \mathcal{G}(g) \circ \alpha_X$. To show that $g = \mathcal{F}(f)$, it suffices to show that $\mathcal{G}(g) = \mathcal{G}\mathcal{F}(f)$ since $\mathcal{G}$ is faithful. Yet, by naturality, $\alpha_Y \circ f = \mathcal{G}\mathcal{F}(f) \circ \alpha_X$; by plugging in the definition of f we have $\alpha_Y \circ \alpha_Y^{-1} \circ \mathcal{G}(g) \circ \alpha_X = \mathcal{G}(g) \circ \alpha_X = \mathcal{G}\mathcal{F}(f) \circ \alpha_X \Rightarrow \mathcal{G}(g) = \mathcal{G}\mathcal{F}(f)$ where the final line follows from the fact that α_X is an isomorphism and can therefore be canceled from both sides. In conclusion, the map $\mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{|bfD}(\mathcal{F}(X), \mathcal{F}(Y))$ is surjective for any X, Y so $\mathcal{F}$ is full.

Finally, $\mathcal{F}$ is obviously essentially surjective; for any object $Y \in \mathbf{D}$, we have an isomorphism $\beta_Y : Y \simeq \mathcal{F}\mathcal{G}(Y)$. Hence the result follows by assigning $X = \mathcal{G}(Y)$.

Next, we will demonstrate the converse result: that if $\mathcal{F}$ is full, faithful, and essentially surjective then it is an equivalence of categories. First, define $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ on objects by sending any object $Y \in \mathbf{D}$ to an element $X \in \mathbf{C}$ with $Y \simeq \mathcal{F}(X)$. Notice that this also gives us a component $\beta_Y : Y \xrightarrow{\sim} \mathcal{F}\mathcal{G}(Y)$ of β.

Next, we will define $\mathcal{G}$ on morphisms to make β natural. Namely, given $g : Y \to Y'$ in $\mathbf{D}$, consider the following commutative-by-definition diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{\;\;\beta_Y\;\;} & \mathcal{F}\mathcal{G}(Y) \\[2pt]
{\scriptstyle g}\Big\downarrow & & \Big\downarrow{\scriptstyle \beta_{Y'} \circ g \circ \beta_Y^{-1}} \\[6pt]
Y' & \xrightarrow[\;\;\beta_{Y'}\;\;]{} & \mathcal{F}\mathcal{G}(Y')
\end{array}
$$

Since $\mathcal{F}$ is full and faithful, there is a unique arrow $\mathcal{G}(Y) \to \mathcal{G}(Y')$ such that $\mathcal{F}\mathcal{G}(h) = \beta_{Y'} \circ g \circ \beta_Y^{-1}$. Define this arrow to be $\mathcal{G}(h)$; then by uniqueness, $\mathcal{G}$ is a functor, and by construction this makes β a natural isomorphism $\mathrm{id}_{\mathbf{D}} \xrightarrow{\sim} \mathcal{F}\mathcal{G}$.

All that remains is to find a natural isomorphism $\alpha : \mathrm{id}_{\mathbf{C}} \xrightarrow{\sim} \mathcal{G}\mathcal{F}$. For this, fix an object $X \in \mathbf{C}$. Then consider $\beta_{\mathcal{F}(X)} : \mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}\mathcal{G}\mathcal{F}(X)$; since $\mathcal{F}$ is full and faithful, there is a unique map $f : X \to \mathcal{G}\mathcal{F}(X)$ with $\mathcal{F}(f) = \beta_{\mathcal{F}(X)}$. In fact, this map is also an isomorphism, since $\mathcal{F}$ is full and faithful, and we define $\alpha_X := f$. The naturality of the α_X follow immediately from the naturality of the $\beta_{\mathcal{F}(X)}$. Hence we are done. $\qquad\square$

3 Adjunctions

We begin with three competing definitions for what an "adjunction" is; we will see that all of these are equivalent, giving us three ways to understand adjunctions. Then we will work with examples.

Definition 6 (Adjunctions with Units and Counits). An *adjunction* between categories $\mathbf{C}$ and $\mathbf{D}$ is a pair of functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ together with natural transformations $\eta : \mathrm{id}_C \to GF$ (called the *unit*) and $\varepsilon : FG \to \mathrm{id}_D$ (called the *counit*) so that for all $X \in C$ and $Y \in D$, the following diagrams commute:

$$
\mathrm{id}_{\mathcal{F}(X)} \left(\; \mathcal{F}(X) \xrightarrow{\mathcal{F}(\eta_X)} \mathcal{F}\mathcal{G}\mathcal{F}(X) \underset{\varepsilon_{\mathcal{F}(X)}}{\longleftarrow} \right. \qquad \mathrm{id}_{\mathcal{G}(Y)} \left(\; \mathcal{G}(Y) \xrightarrow{\eta_{\mathcal{G}(Y)}} \mathcal{G}\mathcal{F}\mathcal{G}(Y) \underset{\mathcal{G}(\varepsilon_Y)}{\longleftarrow} \right.
$$

In this case, we write $F \dashv G$ and say that F and G are *adjoint functors*; more precisely, F is *left adjoint* to G and G is *right adjoint* to F.

Definition 7 (Adjunctions with Just Units). Equivalently, an *adjunction* between categories $\mathbf{C}$ and $\mathbf{D}$ is a pair of functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ together with a natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \to GF$ (called the *unit*) such that the following universal property holds: for any $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, there exists a unique map $\hat{f} : \mathcal{F}(X) \to Y$, called the *adjunct* or *transpose* of f, such that $f = \mathcal{G}(\hat{f}) \circ \eta_X$.

Definition 8 (Adjunctions as Isomorphisms of Hom-Sets). Equivalently, an *adjunction* between categories $\mathbf{C}$ and $\mathbf{D}$ is a pair of functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ together with a bijection

$$
\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(Y))
$$

which is natural in X and Y; i.e., $\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), \_\_) \simeq \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(\_\_))$ and $\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(\_\_), Y) \simeq \mathrm{Hom}_{\mathbf{C}}(\_\_, \mathcal{G}(Y))$ are natural isomorphisms for each X and Y. In this case, the image of $f : \mathcal{F}(X) \to Y$ under the bijection is a map $\hat{f} : X \to \mathcal{G}(Y)$ called the *adjunct* or *tranpose of f*.

This confusing mess is resolved by the following theorem.

Theorem 2 (Equivalence of Definitions). *All three definitions for adjunctions (6, 7, and 8) are equivalent.*

The proof follows from 3 lemmas.

Lemma 3. *Suppose that $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ form an adjunction $\mathcal{F} \dashv \mathcal{G}$ in the sense of Definition 6; that is, there is a unit η and counit ε. Then if $X \in \mathbf{C}$ and $Y \in \mathbf{D}$ and $f : X \to \mathcal{G}(Y)$ is a morphism in $\mathbf{C}$, then there is a unique morphism $\hat{f} : \mathcal{F}(X) \to Y$ in $\mathbf{D}$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{G}\mathcal{F}(X) & & \\
\eta_X \uparrow & \searrow^{\mathcal{G}(\hat{f})} & \\
X & \xrightarrow{f} & \mathcal{G}(Y)
\end{array}
$$

In particular, $\mathcal{F}$ and $\mathcal{G}$ form an adjunction in the sense of Definition 7.

Proof. Define $\hat{f}$ to be the composition map $\mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}\mathcal{G}(Y) \xrightarrow{\varepsilon_Y} Y$; that is, $\hat{f} = \varepsilon_Y \circ \mathcal{F}(f)$. Then $\mathcal{G}(\hat{f}) = \mathcal{G}(\varepsilon_Y) \circ \mathcal{G}\mathcal{F}(f)$, so $\mathcal{G}(\hat{f}) \circ \eta_X = \mathcal{G}(\varepsilon_Y) \circ \mathcal{G}\mathcal{F}(f) \circ \eta_X$. Since η is a natural transformation, this is equal to $\mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)} \circ f$. But $\mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)} = \mathrm{id}_{\mathcal{G}(Y)}$ as in Definition 6, so $\mathcal{G}(\hat{f}) \circ \eta_X = \mathrm{id}_{\mathcal{G}(Y)} \circ f = f$.

Next, we will show that $\hat{f} = \varepsilon_Y \circ \mathcal{F}(f)$ is forced by the condition $\mathcal{G}(\hat{f}) \circ \eta_X = f$. For this, notice that

$$
\mathcal{G}(\hat{f}) \circ \eta_X = f \Rightarrow \mathcal{F}\mathcal{G}(\hat{f}) \circ \mathcal{F}(\eta_X) = \mathcal{F}(f) \Rightarrow \varepsilon_Y \circ \mathcal{F}\mathcal{G}(\hat{f}) \circ \mathcal{F}(\eta_X) = \varepsilon_Y \circ \mathcal{F}(f).
$$

But by naturality, $\varepsilon_Y \circ \mathcal{F}\mathcal{G}(\hat{f}) = \hat{f} \circ \varepsilon_{\mathcal{F}(X)}$, so

$$
\varepsilon_Y \circ \mathcal{F}\mathcal{G}(\hat{f}) \circ \mathcal{F}(\eta_X) = \varepsilon_Y \circ \mathcal{F}(f) \Rightarrow \hat{f} \circ \varepsilon_{\mathcal{F}(X)} \circ \mathcal{F}(\eta_X) = \varepsilon_Y \circ \mathcal{F}(f) \Rightarrow \hat{f} \circ \mathrm{id}_{\mathcal{F}(X)} = \varepsilon_Y \circ \mathcal{F}(f)
$$

where the final implication follows from the commutative diagram in Definition 6. Yet then $\hat{f} = \hat{f} \circ \mathrm{id}_{\mathcal{F}(X)} = \varepsilon_Y \circ \mathcal{F}(f)$ is the desired proof of uniqueness, so we are done. $\square$

Lemma 4. *Suppose that $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ form an adjunction $\mathcal{F} \dashv \mathcal{G}$ in the sense of Definition 7. Then they form an adjunction in the sense of Definition 8.*

Proof. Take $X \in \mathbf{C}$ and $Y \in \mathbf{D}$ and define the map $\phi_{XY} : \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), Y) \to \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(Y))$ by the formula $\phi_{XY}(g) = \mathcal{G}(g) \circ \eta_X$. Recall that by the universal mapping property in Definition 7, for any given $f : X \to \mathcal{G}(Y)$, there is exactly one map $\hat{f}$ such that $f = \phi(\hat{f})$. Hence ϕ_{XY} is a bijection, so we have a bijection $\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), Y) \simeq \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(Y))$. Now, it suffices to show that ϕ_{XY} is natural in X and Y.

First, we will show $\phi_{X\_\_} : \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), \_\_) \simeq \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(\_\_))$ is a natural isomorphism for each X. Therefore, take a map $\psi : Y \to Y'$. The goal is to show that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), Y) & \xrightarrow{\phi_{XY}} & \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(Y)) \\
\psi_* \downarrow & & \downarrow (\mathcal{G}(\psi))_* \\
\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), Y') & \xrightarrow{\phi_{XY'}} & \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(Y'))
\end{array}
$$

Therefore, choose $f : \mathcal{F}(X) \to Y$. The desired result is achieved easily by applying the naturality of η.

$$
\begin{aligned}
(\mathcal{G}(\psi))_*(\phi_{XY}(f)) &= \mathcal{G}(\psi) \circ (\mathcal{G}(f) \circ \eta_X) \\
&= \mathcal{G}(\psi \circ f) \circ \eta_X \\
&= \phi_{XY}(\psi \circ f) \\
&= \phi_{XY}(\psi^*(f)).
\end{aligned}
$$

Lastly, we will show $\phi_{\_Y} : \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(\_\_), Y) \simeq \mathrm{Hom}_{\mathbf{C}}(\_\_, \mathcal{G}(Y))$ is a natural isomorphism for each Y. Therefore, take a map $\psi : X \to X'$. The goal is to show that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X'), Y) & \xrightarrow{\phi_{X'Y}} & \mathrm{Hom}_{\mathbf{C}}(X', \mathcal{G}(Y)) \\
(\mathcal{F}(\psi))^* \downarrow & & \downarrow \psi^* \\
\mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), Y) & \xrightarrow{\phi_{XY}} & \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}(Y))
\end{array}
$$

Therefore, choose $f : \mathcal{F}(X') \to Y$. The desired result is achieved easily by applying the naturality of η.

$$
\begin{aligned}
\psi^*(\phi_{X'Y}(f)) &= (\mathcal{G}(f) \circ \eta_{X'}) \circ \psi \\
&= \mathcal{G}(f) \circ \mathcal{G}\mathcal{F}(\psi) \circ \eta_X \\
&= \mathcal{G}(f \circ \mathcal{F}(\psi)) \circ \eta_X \\
&= \phi_{XY}((\mathcal{F}(\psi)^*)(f)).
\end{aligned}
$$

Hence we are done. $\qquad\square$

Lemma 5. *Suppose that $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ form an adjunction $\mathcal{F} \dashv \mathcal{G}$ in the sense of Definition 8. Then they form an adjunction in the sense of Definition 6.*

Proof. Consider the bijection $\phi_{X\mathcal{F}(X)} : \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}(X), \mathcal{F}(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(X, \mathcal{G}\mathcal{F}(X))$; define η_X to be the image (the transpose) of the identity map $\mathrm{id}_{\mathcal{F}(X)}$. Similarly, consider the bijection $\phi_{\mathcal{G}(Y)Y} : \mathrm{Hom}_{\mathbf{D}}(\mathcal{F}\mathcal{G}(Y), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(\mathcal{G}(Y), \mathcal{G}(Y))$; define ε_Y to be the preimage (the transpose) of the identity map $\mathrm{id}_{\mathcal{G}(Y)}$.

First, we will show that both η and ε are natural using the fact that ϕ_{XY} is natural in X and Y. For this, it is worth recounting what "naturality in X and Y" means. Explciitly, naturality in X means that given a map $g : X \to X'$, and any map $f : X' \to \mathcal{G}(Y)$, $\phi_{XY}^{-1}(f \circ g) = \phi_{X'Y}^{-1}(f) \circ \mathcal{F}(g)$. Similarly, naturality in Y means that given a map $g : Y \to Y'$, and any map $f : \mathcal{F}(X) \to Y$, $\phi_{XY'}(g \circ f) = \mathcal{G}(g) \circ \phi_{XY}(f)$.

Then, suppose that we have a map $\varphi : X \to X'$. We wish to show that the following square commutes:

$$X \xrightarrow{\eta_X} \mathcal{GF}(X)$$
$$\varphi \downarrow \qquad \qquad \downarrow \mathcal{GF}(\varphi)$$
$$X' \xrightarrow[\eta_{X'}]{} \mathcal{GF}(X')$$

Firstly, notice that by the naturality in Y of ϕ_{XY}, we have (dropping the XY in ϕ_{XY} for convenience),

$$\phi(\mathcal{F}(\varphi)) = \phi(\mathcal{F}(\varphi) \circ \mathrm{id}_{\mathcal{F}(X)}) = \mathcal{GF}(\varphi) \circ \phi(\mathrm{id}_{\mathcal{F}(X)}) = \mathcal{GF}(\varphi) \circ \eta_X.$$

On the other hand, by the naturality in X of ϕ_{XY}, we have $\phi^{-1}(\eta_{X'} \circ \varphi) = \phi^{-1}(\eta_{X'}) \circ \mathcal{F}(\varphi) = \mathrm{id}_{X'} \circ \mathcal{F}(\varphi) = \mathcal{F}(\varphi)$; applying ϕ again, we have $\eta_{X'} \circ \varphi = \phi(\mathcal{F}(\varphi)) = \mathcal{GF}(\varphi) \circ \eta_X$, proving the desired naturality. One can dually prove the naturality of ε by citing naturality in X and then naturality in Y.

Finally, it remains to show that η and ε satisfy the commutative diagrams outlined in Definition 6. Namely, we must have $\varepsilon_{\mathcal{F}(X)} \circ \mathcal{F}(\eta_X) = \mathrm{id}_{\mathcal{F}(X)}$ and $\mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)} = \mathrm{id}_{\mathcal{G}(Y)}$. For this, notice that by naturality in X,

$$\mathrm{id}_{\mathcal{F}(X)} = \phi^{-1}(\phi(\mathrm{id}_{\mathcal{F}(X)})) = \phi^{-1}(\eta_X) = \phi^{-1}(\mathrm{id}_{\mathcal{GF}(X)} \circ \eta_X) = \phi^{-1}(\mathrm{id}_{\mathcal{GF}(X)}) \circ \mathcal{F}(\eta_X) = \varepsilon_{\mathcal{F}(X)} \circ \mathcal{F}(\eta_X).$$

Similar reasoning holds for the other equality; namely, by the naturality of ϕ_{XY} in Y,

$$\mathrm{id}_{\mathcal{G}(Y)} = \phi(\phi^{-1}(\mathrm{id}_{\mathcal{G}(Y)})) = \phi(\varepsilon_Y) = \phi(\varepsilon_Y \circ \mathrm{id}_{\mathcal{F}(X)}) = \mathcal{G}(\varepsilon_Y) \circ \phi(\mathrm{id}_{\mathcal{FG}(Y)}) = \mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)}.$$

$\square$

Together, these three lemmas prove Theorem 2.

Broadly speaking, it is easier to think of adjunctions using the definition in terms of natural isomorphisms of hom-sets. However, it is usually easier to verify that constructions form adjunctions using the unit and counit or unit definitions. Therefore, all three definitions are genuinely helpful in different situations.

3.1 Examples of Adjunctions

Example 3 (Product-Hom Adjunction). Define $\mathcal{F} : \mathbf{Sets} \to \mathbf{Sets}$ by $\mathcal{F}(X) = X \times Z$ and $\mathcal{G} : \mathbf{Sets} \to \mathbf{Sets}$ by $\mathcal{G}(Y) = \mathrm{Hom}(Z, Y)$ for a fixed set Z, with the obvious operations on morphisms (which in this case are just set-theoretic functions). We will show that there exists an adjunction $\mathcal{F} \dashv \mathcal{G}$ using two definitions, to illustrate how the isomorphism-of-hom sets definition can be more enlightening, but proving the conditions for the adjunction is easier with the definition using the unit and/or the counit.

Isomorphism of Hom-Sets: Define a map $\phi : \mathrm{Hom}(X \times Z, Y) \to \mathrm{Hom}(X, \mathrm{Hom}(Z, Y))$ as follows: given a map $f : X \times Z \to Y$, define $\phi(f) : X \to \mathrm{Hom}(Z, Y)$ given by sending x to the map $z \mapsto f(x, z)$. To see why ϕ is a bijection, consider $\psi : \mathrm{Hom}(X, \mathrm{Hom}(Z, Y)) \to \mathrm{Hom}(X \times Z, Y)$ given by sending $g : X \to \mathrm{Hom}(Z, Y)$ to the map $(x, z) \mapsto g(x)(z)$. One may easily verify that ϕ and ψ are two-way inverses, so ϕ is bijective.

Next, we must show that ϕ is natural is X and Y. Namely, take a map $\rho : Y \to Y'$. The goal is to show that the following diagram is commutative:

$$\mathrm{Hom}(X \times Z, Y) \xrightarrow{\phi_{XY}} \mathrm{Hom}(X, \mathrm{Hom}(Z, Y))$$
$$\rho_* \downarrow \qquad \qquad \downarrow (\mathcal{G}(\rho))_*$$
$$\mathrm{Hom}(X \times Z, Y') \xrightarrow[\phi_{XY'}]{} \mathrm{Hom}(X, \mathrm{Hom}(Z, Y'))$$

Now choose $f : X \times Z \to Y$. Then $\rho_*(f) = \rho \circ f$, so $\phi_{XY'}(\rho_*(f))$ is the map $X \to \mathrm{Hom}(X, Y)$ given by sending x to the map $z \mapsto \rho(f(x, z))$. On the other hand, $\phi_{XY}(f)$ is the map given by sending x to the map $z \mapsto f(x, z)$. Then, $\mathcal{G}(\rho)$ is the map $\mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, Y')$ given by sending f to $\rho \circ f$, so $\mathcal{G}(\rho)_*(\phi_{XY}(f))$ is the map given by sending x to the map $z \mapsto f(x, z)$, and then sending that map to its composition with ρ; in summary, $\mathcal{G}(\rho)_*(\phi_{XY}(f))$ is the map given by sending x to the map $z \mapsto \rho(f(x, z))$, exactly the same

as $\phi_{XY'}(\rho_*(f))$. Hence we have shown naturality in Y.

Similarly, take a map $\rho : X \to X'$. The goal is to show that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}(X' \times Z, Y) & \xrightarrow{\phi_{X'Y}} & \mathrm{Hom}(X, \mathrm{Hom}(Z, Y)) \\ {\scriptstyle \mathcal{F}(\rho)^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho^*} \\ \mathrm{Hom}(X \times Z, Y) & \xrightarrow{\phi_{XY}} & \mathrm{Hom}(X, \mathrm{Hom}(Z, Y)) \end{array}$$

Now choose $f : X' \times Z \to Y$. Then $\mathcal{F}(\rho)$ is the obvious map $X \times Z \to X' \times Z$ given by $(x, z) \mapsto (\rho(x), z)$, so $\mathcal{F}(\rho)^*(f) = f \circ \mathcal{F}(\rho)$. Then $\phi_{XY}(\mathcal{F}(\rho^*(f))) = \phi_{XY}(f \circ \mathcal{F}(\rho))$ is the map given by sending x to the map $z \mapsto f(\mathcal{F}(\rho)(x, z)) = f(\rho(x), z)$. On the other hand, $\phi_{X'Y}(f)$ is the map given by sending x' to the map $z \mapsto f(x', z)$, so $\rho^*(\phi_{X'Y}(f)) = \rho \circ \phi_{X'Y}(f)$ is the map $z \mapsto f(\rho(x), z)$, exactly as desired. Hence we have shown naturality in X, so we have shown naturality in both X and Y, so we are done.

Unit: Define a natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \to \mathcal{GF}$ by defining $\eta_X : X \to \mathrm{Hom}(Z, X \times Z)$ to be given by $x \mapsto (z \mapsto (x, z))$. First, we must show that η_X is natural. Therefore, choose $\rho : X \to X'$. Then

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathrm{Hom}(Z, X \times Z) \\ {\scriptstyle \rho}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{GF}(\rho)} \\ X' & \xrightarrow{\eta_{X'}} & \mathrm{Hom}(Z, X' \times Z) \end{array}$$

must commute for η to be a natural transformation. Now, $\mathcal{GF}(\rho) \circ \eta_X$ is the map given by sending x to the map $\mathcal{GF}(\rho)(z \mapsto (x, z)) = z \mapsto \mathcal{F}(\rho)(x, z) = z \mapsto (\rho(x), z)$. Yet $\eta_{X'} \circ \rho$ is the map given by sending x to $\rho(x)$, and then sending $\rho(x)$ to the map $z \mapsto (\rho(x), z)$. In other words, both $\mathcal{GF}(\rho) \circ \eta_X$ and $\eta_{X'} \circ \rho$ are given by sending x to $z \mapsto (\rho(x), z)$. Therefore η is indeed a natural transformation. To conclude the proof that $\mathcal{F} \dashv \mathcal{G}$, it suffices to show that η has the UMP in Definition 7.

Take sets X, Y, and a morphism $f : X \to \mathrm{Hom}(Z, Y)$. We seek to show that there exists a unique map $\hat{f} : X \times Z \to Y$ such that $\mathcal{G}(\hat{f}) \circ \eta_X = f$. This follows from the below equivalences:

$$\mathcal{G}(\hat{f}) \circ \eta_X = f \Leftrightarrow \mathcal{G}(\hat{f}) \circ \eta_X(x) = f(x) \text{ for all } x \in X \Leftrightarrow \mathcal{G}(\hat{f})(z \mapsto (x, z)) = f(x) \Leftrightarrow z \mapsto \hat{f}(x, z) = f(x).$$

That is, $\hat{f} : X \times Z \to Y$ is uniquely determined by the condition $\hat{f}(x, z) = f(x)(z)$, and in this case we indeed have $\mathcal{G}(\hat{f}) \circ \eta_X = f$. Therefore η has the desired UMP, and we are done.

Example 4 (Free-Forgetful Adjunction)**.** This particular adjunction appears in numerous forms: between the category of sets and vector spaces, the category of sets and groups, even the category of graphs and the category of (small) categories; here, we will focus on the specific adjunction between **Sets** and $\mathbf{Vect}_k$. Namely, the functor $\mathcal{F} : \mathbf{Sets} \to \mathbf{Vect}_k$ is given by sending a set S to the vector space $\mathcal{F}(S)$ freely generated over S; that is, the set of all finite formal k-linear combinations of elements of S, with the obvious vector space structure. The functor $\mathcal{G} : \mathbf{Vect}_k \to \mathbf{Sets}$ is given by sending a vector space V to its underlying set, "forgetting" the vector space structure. We will use the unit definition to show that $\mathcal{F} \dashv \mathcal{G}$.

Define a natural transformation $\eta : \mathrm{id}_{\mathbf{Sets}} \to \mathcal{GF}$ by defining $\eta_X : X \to \mathcal{GF}(X)$ to be the natural injection; that is, sending $x \in X$ to the basis element $1x \in \mathcal{F}(X)$, and then to the element $1x = \mathcal{G}(1x)$. First, we must show that η is natural; choose $\rho : X \to X'$. Then $\mathcal{GF}(\rho)$ is the set-theoretic function which underlies the k-linear map $\mathcal{F}(\rho)$ given by extending ρ k-linearly from the basis X of $\mathcal{F}(X)$. In particular, $\mathcal{GF}(\rho) \circ \eta_X$ is the map given by $x \overset{\eta_X}{\mapsto} 1x \overset{\mathcal{GF}(\rho)}{\mapsto} 1\rho(x)$. On the other hand, $\eta_{X'} \circ \rho$ is the map given by $x \overset{\rho}{\mapsto} \rho(x) \overset{\eta_{X'}}{\mapsto} 1\rho(x)$. Hence $\mathcal{GF}(\rho) \circ \eta_X = \eta_{X'} \circ \rho$, so η is natural, as desired.

To conclude the proof that $\mathcal{F} \dashv \mathcal{G}$, it suffices to show that η has the UMP in Definition 7. Take a space X and a vector space V, and a morphism $f : X \to \mathcal{G}(V)$. We seek to show that there exists a unique morphism $f : \mathcal{F}(X) \to V$ such that $\mathcal{G}(\hat{f}) \circ \eta_X = f$. This statement is obvious when written out in plain English: it

amounts to the fact that given a map f of the basis X of the vector space $\mathcal{F}(X)$ into another vector space V, there is a unique k-linear map $\hat{f}$ extending v to all of $\mathcal{F}(X)$. Hence we are done.

Example 5 (Tensor-Hom Adjunction)**.** The final adjunction which we will examine is called the tensor-hom adjunction, and is very important in homological algebra (it is the foundation of the relationship between the Ext and Tor functions, for example). It is certainly the most difficult of the examples to understand, so do not feel bad if you need to skip it; the first two examples are plenty to get the idea.

Let us begin by recounting the universal property which defines the tensor product up to unique isomorphism (I will not prove this; it is proven in any good book on algebra, such as Dummit and Foote). Let A be a commutative ring (with unity), and X and Y be A-modules. Then the tensor product $X \otimes_A Y$ and the natural map $\varphi : X \times Y \to X \otimes_A Y$ (given by $(x, y) \mapsto x \otimes y$) satisfy the following universal property: given any A-bilinear map $f : X \times Y \to Z$, there exists a unique A-linear map $\widetilde{f} : X \otimes Y \to Z$ such that $f = \widetilde{f} \circ \varphi$.

Now let A and B be commutative rings. Recall that $\mathbf{Mod}_A$ is the category of A-modules (and similarly $\mathbf{Mod}_B$ is the category of B-modules). Fix an (A, B)-module Z. Then define the functor $\mathcal{F} : \mathbf{Mod}_A \to \mathbf{Mod}_B$ by $\mathcal{F}(X) = X \otimes_A Z$ (with the obvious operation on morphisms). Similarly, define the functor $\mathcal{G} : \mathbf{Mod}_B \to \mathbf{Mod}_A$ given by $\mathcal{F}(Y) = \mathrm{Hom}_B(Z, Y)$. Then we have an adjunction $\mathcal{F} \dashv \mathcal{G}$.

This adjunction is best understood as a isomorphism

$$\mathrm{Hom}_B(X \otimes_A Z, Y) \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_B(Y, Z))$$

natural in X and Y. However, proving that it is an adjunction is especially easy using the unit/counit definition of adjunctions, so this gives a chance to show the utility of our third definition of adjunctions.

Define the unit $\eta : \mathrm{id}_{\mathbf{Mod}_A} \to \mathcal{G}\mathcal{F}$ as follows: given an A-module X, let the map $\eta_X : X \to \mathrm{Hom}_B(Z, X \otimes_A Z)$ be given by sending $x \in X$ to the B-module homomorphism given by $\eta_X(x)(z) = x \otimes z$. Similarly, we define the unit $\varepsilon : \mathcal{F}\mathcal{G} \to \mathrm{id}_{\mathbf{Mod}_B}$ as follows: given a B-module Y, let the map $\varepsilon_Y : \mathrm{Hom}_B(Z, Y) \otimes_A Z \to Y$ be given by evaluation; that is, given $\phi \otimes z$, $\varepsilon(\phi \otimes z) = \phi(z)$ (and we linearly extend since the simple tensors span the entire tensor product). It is left as an exercise to the reader to prove that η and ε are indeed natural.

Now, the unit and counit commutativity conditions can be computed easily. Namely, given a A-module X, to show that the map $\varepsilon_{\mathcal{F}(X)} \circ \mathcal{F}(\eta_X) : X \otimes_A Z \to \mathrm{Hom}_B(Z, X \otimes_A Z) \otimes_A Z \to X \otimes_A Z$ is equal to $\mathrm{id}_{\mathcal{F}(X)}$, it suffices to show the result on simple tensors of $X \otimes_A Z$. Yet we immediately have $\varepsilon_{\mathcal{F}(X)} \circ \mathcal{F}(\eta_Y)(x \otimes z) = \eta_X(x)(z) = x \otimes z$, exactly as desired. Similarly, the map $\mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)} : \mathrm{Hom}_B(Z, Y) \to \mathrm{Hom}_B(Z, \mathrm{Hom}_B(Z, Y) \otimes_A Z) \to \mathrm{Hom}_B(Z, Y)$ is computed as so: given $\phi \in \mathrm{Hom}_S(X, Z)$, $\mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)}(\phi)$ is the map defined by $\mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)}(z) = \varepsilon_Y(\phi \otimes z) = \phi(z)$, whence $\mathcal{G}(\varepsilon_Y) \circ \eta_{\mathcal{G}(Y)}(\phi) = \phi$, as desired.